OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

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CHAPTER-17 : THE HYPERBOLA

UNIT TEST-1

1. Find the condition on 'a' and 'b' for which two distinct chords of the hyperbola $\frac{x^2}{2a^2}-\frac{y^2}{2b^2}=1$ passing through (a, b) are bisected by the line $x + y = b$.

2. Prove that the length of the tangent at any point of hyperbola intercepted between the point of contact and the transverse axis is the harmonic mean between the lengths of perpendiculars drawn from the foci on the normal at the same point.

3. A normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N and lines MP and NP are drawn perpendicular to the axes meeting at P. Prove that the locus of P is the hyperbola $a^2x^2-b^2y^2=\left(a^2+b^2\right)^2$.

4. The perpendicular from the centre on the normal at any point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meet at R. Find the locus of R.

5. If the normals at (x, y); $r = 1, 2, 3, 4$ on the rectangular hyperbola $xy = c^2$ meet at the point $Q(h, k)$, prove that the sum of the ordinates of the four points is k. Also prove that the product of the ordinates is $-C^4$.

6. If $H\ (x, y) = 0$ represents the equation of a hyperbola and $A\ (x, y) = 0$, $C(x, y) = 0$ the joint equation of its asymptotes and the conjugate hyperbola respectively, then for any point (α, β) in the plane, $H\ (\alpha, \beta)$, $A(\alpha, \beta)$, and $C(\alpha, \beta)$ are in A.P. Prove.

7. Let $P\ (a \sec\alpha,\ b\tan\alpha)$ and $Q(a\sec\beta,\ b\tan\beta)$, where $\alpha+\beta=\frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If (h, k) is the point of intersection of the normals at P and Q, then Find k.

8. If two points P and Q on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose centre is C, are such that CP is perpendicular to $CQ, a<b,$ then prove that $\frac{1}{(CP)^2}+\frac{1}{(CQ)^2}=\frac{1}{a^2}-\frac{1}{b^2}$

Hints and Solutions

1. Consider a point on the line $x + y = b$ and then find a chord with this point as the mid-point. Then substitute the point in the equation of the chord to get the condition between 'a' and 'b'.

Let the line $x + y = b$ bisect the chord at $P(\alpha, b - \alpha)$

$\therefore$ Equation of the chord whose mid-point is $P(\alpha, b - \alpha)$ is:

$$\frac{x\alpha}{2a^2}-\frac{y(b-\alpha)}{2b^2}=\frac{\alpha^2}{2a^2}-\frac{(b-\alpha)^2}{2b^2}$$

Since it passes through (a, b)

$$\therefore \quad \frac{\alpha}{2a}-\frac{(b-\alpha)}{2b}=\frac{\alpha^2}{2a^2}-\frac{(b-\alpha)^2}{2b^2}$$

$$\alpha^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\alpha\left(\frac{1}{b}-\frac{1}{a}\right)=0\Rightarrow a=b$$

2. Proceed according to the question to prove the above statement.

$$\frac{P_1}{P}=\frac{S_1G}{TG}=\frac{e^2x_1-ae}{e^2x_1-a\cos\theta}=\frac{ae^2-ae\cos\theta}{ae^2-a\cos^2\theta}$$

$$\therefore \quad \frac{P_1}{P} = \frac{e(e-\cos\theta)}{(e-\cos\theta)(e+\cos\theta)}$$

$$\therefore \quad \frac{P}{P_1} = \frac{e+\cos\theta}{e} = 1 + \frac{\cos\theta}{e}$$

Similarly we get $\frac{P}{P_1} = 1 - \frac{\cos\theta}{e}$

$$\therefore \frac{P}{P_1} + \frac{P}{P_2} = 2 \Rightarrow \frac{1}{P_1} + \frac{P}{P_2} = \frac{2}{P}$$

Hence Proved.

3. Find the co-ordinates of the point M and N and then eliminate the parameter between the ordinate and abscissae.

The equation of normal at the point $Q(a\sec\phi, b\tan\phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is as

$$\cos\phi + by\cot\phi = a^2 + b^2 \quad \text{...(i)}$$

The normal (i) meets the x-axis in $M\left(\frac{a^2+b^2}{a}\sec\varphi, 0\right)$ and y-axis in $N\left(0, \frac{a^2+b^2}{b}\tan\varphi\right)$

$\therefore$ Equation of MP, the line through M and perpendicular to axis, is

$$x = \left(\frac{a^2+b^2}{a}\right)\sec\phi \quad \text{or} \quad \sec\phi = \frac{ax}{(a^2+b^2)} \quad \text{...(ii)}$$

and the equation of NP, the line through N and perpendicular to the y-axis is

$$y = \left(\frac{a^2+b^2}{b}\right)\tan\phi \quad \text{or} \quad \tan\phi = \frac{by}{(a^2+b^2)} \quad \text{...(iii)}$$

The locus of the point is the intersection of MP and NP and will be obtained by eliminating ϕ from (ii) and (iii), so we have $\sec^2\phi - \tan^2\phi = 1$

$$\Rightarrow \quad \frac{a^2x^2}{(a^2+b^2)^2} - \frac{b^2y^2}{(a^2+b^2)^2} = 1$$

or $\quad a^2x^2 - b^2y^2 = (a^2+b^2)^2$

is the required locus of P.

4. Solve the equation of the normal and the equation of line perpendicular to it passing through the origin. Let (x_1, y_1) be any point on the hyperbola.

So, $\quad \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$

The equation of the normal at (x_1, y_1) is $\frac{x-x_1}{\frac{x_1}{a^2}} = \frac{y-y_1}{-\frac{y_1}{b^2}}$

or $\quad \frac{x_1}{a^2}(y-y_1) + \frac{y_1}{b^2}(x-x_1) = 0 \quad \text{(ii)}$

'm' of the normal $= -\frac{a^2y_1}{b^2x_1}$

$\therefore$ The equation of the perpendicular from the centre (0, 0) on (ii) is

$$y = \frac{b^2x_1}{a^2y_1}\cdot x \quad \text{...(iii)}$$

The intersection of (ii) and (iii) is R and the required locus is obtained by eliminating x_1, y_1 from (i), (ii) and (iii).

From (iii), $\frac{x_1}{a^2y} = \frac{y_1}{b^2x} = t$ (say)

Putting in (ii), $yt(y - b^2xt) + xt(x - a^2yt) = 0$

or $\quad (x^2+y^2)t - (a^2+b^2)xyt^2 = 0$

But $t \to 0$ for then $(x_1, y_1) = (0, 0)$ which is not true.

$$\therefore \quad t = \frac{x^2+y^2}{xy(a^2+b^2)};$$

$$\therefore x_1 = \frac{x^2+y^2}{xy(a^2+b^2)}a^2y = \frac{a^2(x^2+y^2)}{x(a^2+b^2)}$$

and $\quad y_1 = \frac{x^2+y^2}{xy(a^2+b^2)}b^2x = \frac{b^2(x^2+y^2)}{y(a^2+b^2)}$

$$\therefore \text{ from (i), } \frac{1}{a^2}\cdot\frac{a^4(x^2+y^2)^2}{x^2(a^2+b^2)} - \frac{1}{b^2}\cdot\frac{b^4(x^2+y^2)^2}{y^2(a^2+b^2)^2} = 1$$

or $\quad \{x^2+y^2)^2.\left(\frac{a^2}{x^2} - \frac{b^2}{y^2}\right) = (a^2+b^2)^2.$

5. Write the equation of the normal in the parametric from and then use the theory of equations. Any point on the curve $xy = c^2$ is $\left(ct, \frac{c}{t}\right)$

The equation of the normal to the hyperbola at the point $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = \frac{-1}{\left(\frac{dy}{dx}\right)_{ct,\frac{c}{t}}}(x - ct).$$

Here, $\quad xy = c^2$; or $y = \frac{c^2}{x}$, $\quad \therefore \quad \frac{dy}{dx} = \frac{-c^2}{x^2}$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{ct,\frac{c}{t}} = \frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

$\therefore$ The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is

$$\frac{c}{t} = t^2(x - ct) \text{ or } ty - c = t^3(x - ct)$$

or $\quad ct^4 - t^3x + ty - c = 0$

The normal passes through (h, k). So

$$ct^4 - t^3h + tk - c = 0$$

Let the roots of (i) be t_1, t_2, t_3, t_4. Then $x_r = ct, y_r = \frac{c}{t_r}$

$\therefore$ Sum of ordinates $= y_1 + y_2 + y_3 + y_4$

$$= \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = c\frac{t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3}{t_1t_2t_3t_4}$$

$= c \cdot \frac{-k/c}{-c/c} = k$, {from roots of the equation (i) and, product of the ordinates}

$$= y_1y_2y_3y_4 = \frac{c}{t_1}\cdot\frac{c}{t_2}\cdot\frac{c}{t_3}\cdot\frac{c}{t_4} = \frac{c^4}{t_1t_2t_3t_4} = \frac{c^4}{-c/c} = -c^4$$

6. Let $H(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1.$

Then, $\quad A(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

and $\quad C(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1$

We observe that

$$2A(\alpha, \beta) = H(\alpha, \beta) + C(\alpha, \beta)$$

Hence, $H(\alpha, \beta), A(\alpha, \beta)$ and $C(\alpha, \beta)$ are in A.P.

7. Equation of normal at $P(a \sec \alpha, b \tan \alpha)$ and $Q(a \sec \beta, b \tan \beta)$ are

$$ax \cos \alpha + by \cot \alpha = a^2 + b^2 \quad \text{...(i)}$$

$$ax \cos \beta + by \cot \beta = a^2 + b^2$$

$$ax \cos\left(\frac{\pi}{2} - \alpha\right) + by \cot\left(\frac{\pi}{2} - \alpha\right) = a^2 + b^2$$

$\Rightarrow \quad ax \sin\alpha + by \tan\alpha = a^2 + b^2 \quad$ ...(ii)

Equation (1) and (2) pass through (h, k) then

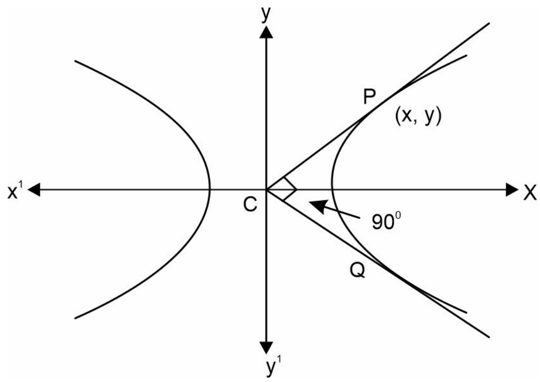
$ah \cos \alpha + bk \cot \beta = a^2 + b^2$ and $ah \sin \alpha + bk \tan \alpha = a^2 + b^2$

solving above two equations for k, we get

$$k = -\left(\frac{a^2 + b^2}{b}\right)$$

8. Let $P(x, y)$ be any point on the given hyperbola. Let slope of CP is m, then equation of CP is $y = mx$ Solving

$y = mx$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \frac{x^2}{a^2} - \frac{m^2x^2}{b^2} = 1$



$\therefore \quad x^2 = \frac{a^2b^2}{(b^2 - a^2m^2)}$

and $\quad y^2 = m^2x^2 = \frac{a^2m^2b^2}{b^2 - a^2m^2}$

$\therefore \quad x^2 + y^2 = \frac{a^2b^2}{(b^2 - a^2m^2)}(1 + m^2)$

$\therefore \quad (CP)^2 = \frac{a^2b^2(1 + m^2)}{(b^2m^2 - a^2)}$

$\Rightarrow \quad \frac{1}{(CP)^2} = \frac{b^2 - a^2m^2}{a^2b^2(1 + m^2)} \quad$ ...(1)

and equation of CQ is

$y = -\frac{1}{m}x$ replacing m by $-\frac{1}{m}$ in (1)

then $\frac{1}{(CQ)^2} = \frac{b^2 - a^2\left(-\frac{1}{m}\right)^2}{a^2b^2\left(1 + \left(-\frac{1}{m^2}\right)^2\right)} = \frac{b^2m^2 - a^2}{a^2b^2(1 + m^2)}$ ...(2)

Adding (1) and (2) then

$$\frac{1}{(CP)^2} + \frac{1}{(CQ)^2} = \frac{b^2(1 + m^2) - a^2(1 + m^2)}{a^2b^2(1 + m^2)}$$

$$= \frac{b^2 - a^2}{a^2b^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$